Real Analysis I (MA2108)

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Chapter 1

Thm 1.1.1 $\sqrt{2}$ is an irrational number

Thm 1.2.1 (Well-Ordering Property of $\mathbb{N}$) Every nonempty subset S of $\mathbb{N}$ has a least element, i.e. $\exists m \in S$ s.t. $m \leq n$ for all $n \in S$

Thm 1.2.4 General mathematical Induction:
Let $n_0 \in \mathbb{N}$. Suppose that

1. $P(n_o)$ is true; and
2. for each natural number $k \geq n_0, P(k)$ is true $\longrightarrow P(k+1)$ is true.

Then $P(n)$ is true for all natural numbers $n \geq n_0$.

Thm 1.3.1 Algebraic properties of $\mathbb{R}$

1. The binary operation **addition** satisfies Commutativity (A1), Associativity (A2), Existence of zero element (A3), Existence of inverse (A4)
2. The binary operation **multiplication** satisfies Commutativity (M1), Associativity (M2), Existence of zero element (M3), Existence of inverse (M4)
3. (D) Distributivity of multiplication over addition: $a \times (b+c) = a \times b + a \times c$.

Because of the properties (A1) - (A4), (M1) - (M4), and (D), we say that $(\mathbb{R}, +, \cdot)$ forms a **field**.

Remark: $\mathbb{Q}$ forms a field, but $\mathbb{Z}$ and $\mathbb{N}$ do not.

Thm 1.3.2. Let $a, b, c \in \mathbb{R}$

1. *(Uniqueness of additive inverse)*. $a + b = 0 \rightarrow b = -a$
2. *(Uniqueness of multiplicative inverse)*. $((a \neq 0) \wedge (a \cdot b = 1)) \rightarrow b = \frac{1}{a}$
3. $a + b = b \rightarrow a = 0$
4. $(b \neq 0 \wedge a \cdot b = b) \rightarrow a = 1$
5. $a \cdot 0 = 0$
6. $a \cdot b = 0 \rightarrow a = 0 \vee b = 0$
7. *(Cancellative property)* $(a \neq 0 \wedge (a \cdot b = a \cdot c)) \rightarrow b = c$

Thm 1.3.3 (Order Properties of $\mathbb{R}$). Let $a, b, c, d \in \mathbb{R}$. A binary relation $>$ on $\mathbb{R}$ satisfies

1. (O1) $a > b \rightarrow a + c > b + c$
2. (O2) $((a > 0) \wedge (b > 0)) \rightarrow a \cdot b > 0$

3. (O3) (Trichotomy Property). Exactly one of the following holds: $a > b,\ a = b,\ b > a$
4. (O4) (Transitive Property). $((a > b) \wedge (b > c)) \to a > c$

Thm 1.3.4. Let $a, b, c \in \mathbb{R}$, then:

1. $a > b \Leftrightarrow a - b > 0$. In particular $c < 0 \Leftrightarrow -c > 0$.
2. Exactly one of the following holds: $a > 0,\ a = 0,\ b > 0$
3. If $a > b$ and $c > 0$, then $c \cdot a > c \cdot b$. If $a > b$ and $c < 0$, then $c \cdot a < c \cdot b$
4. If $a \geq b$ and $b \geq a$, then $a = b$

Thm 1.3.6.

1. If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$
2. $1 > 0$
3. If $n \in \mathbb{N}$, then $n > 0$.
4. If $a > 0$, then $\frac{1}{a} > 0$

Thm 1.3.7 If $a \in \mathbb{R}$ s.t. $0 \leq a < \epsilon$ for every positive number ϵ, then $a = 0$.

Remark. If $a, b \in \mathbb{R}$ s.t. $a - \epsilon \leq b$ for every $\epsilon > 0$, then $a \leq b$.

Bernoulli Ineq If $x \geq -1$, then $(1 + x)^n \geq 1 + nx \quad \forall n \in \mathbb{N}$

Proof: Usual induction

AM/GM/HM Proof: Forward-backward induction.

Thm 1.6.1 (Properties of absolute value). For all $a, b, c \in \mathbb{R}$

1. $|a| \geq 0$, $a \leq |a|$ and $-a \leq |a|$
2. $|a| = 0 \iff a = 0$
3. $|-a| = |a|$
4. $|ab| = |a| \cdot |b|$
5. $|a|^2 = a^2$
6. If $c \geq 0$, then $|a| \leq c \iff -c \leq a \leq c$
7. $-|a| \leq a \leq |a|$

Thm 1.6.2 (Triangle Ineq).

$$|a| - |b| \leq \big||a| - |b|\big| \leq |a \pm b| \leq |a| + |b|$$

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2 The Completeness of Real Numbers

2.1 Boundedness

Definition 2.1.1. A non-empty set of real numbers $S \subseteq \mathbb{R}$ is said to be **bounded above** if there exists some $M \in \mathbb{R}$ such that

$$x \leq M \quad \forall x \in S$$

Definition 2.1.3. A non-empty set is said to be **bounded** if it is bounded above and bounded below.

Remarks: Upper bounds and lower bounds are not unique.

2.2 Maximum & Minimum

The **maximum** of a non-empty set of real numbers $S \subseteq \mathbb{R}$ is the **unique** upper bound M that satisfies $M \in S$.

2.3 Supremum & Infimum

The **supremum** of a non-empty set of real numbers $S \subseteq \mathbb{R}$ is the **unique** i) upper bound M that satisfies the property that ii) if M' is an upper bound of S, then $M' \geq M$.

Lemma 2.3.2. $M = \sup S \iff \forall \epsilon > 0, \exists x_\epsilon \in S$ s.t. $x_\epsilon > M - \epsilon$.

Remark: If S has a maximum, then sup S = max S.

The **infimum** of a non-empty set of real numbers $S \subseteq \mathbb{R}$ is the **unique** i) lower bound m that satisfies the property that ii) if m' is an lower bound of S, then $m' \leq m$.

Lemma 2.3.6. $M = \inf S \iff \forall \epsilon > 0, \exists x_\epsilon \in S$ s.t. $x_\epsilon < m + \epsilon$.

2.3.7 Completeness Property of $\mathbb{R}$

Every **non-empty** subset of $\mathbb{R}$ which is **bounded above** (below) has a supremum (infimum) in $\mathbb{R}$.

2.4 Corollary of the completeness property

Thm 2.4.1 (Archimedean property of $\mathbb{R}$) For any $x \in \mathbb{R}$, there exists $n_x \in \mathbb{N}$ s.t. $x < n_x$.

Equivalently, we can say $\mathbb{N}$ is not bounded above in $\mathbb{R}$

Corollary: $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$. In particular if $\epsilon > 0$, then there exists $n_\epsilon \in \mathbb{N}$ s.t. $0 < \frac{1}{n_\epsilon} < \epsilon$.

Thm 2.4.5 (Existence of the positive k-th root of a positive real number). Let $c > 0, k \in \mathbb{N}$. Then there exists a unique, positive real number a s.t. $a^k = c$.

Thm 2.4.6 (Density Theorem) For any $x, y \in \mathbb{R}$ s.t. $x < y$, there exists a rational number $q \in \mathbb{Q}$ s.t. $x < q < y$.

Proof: consequence of Archimedean property → $\exists n$ s.t. $y - x > \frac{1}{n}$.

Corollary: Let $\alpha \in \mathbb{R}$ and $E = \{x \in \mathbb{Q} : x < \alpha\} \subseteq \mathbb{Q}$, then sup $E = \alpha$.

Def 2.4.9. A subset D of $\mathbb{R}$ is said to be **dense** in $\mathbb{R}$ if for any $a, b \in \mathbb{R}$ with $a < b$, $D \cap (a, b) \neq \emptyset$

Fact: $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{Q}$ are dense in $\mathbb{R}$.

2.5 More on Intervals

Definition. An interval is a subset I of $\mathbb{R}$ s.t. if $x < t < y$ and $x, y \in I$, then $t \in I$.

3 Sequences

3.1 Preliminary definitions

Definition. a **sequence** in $\mathbb{R}$ is a real-valued function $X : \mathbb{N} \to \mathbb{R}$.

Definition. Let $a \in \mathbb{R}$ and $\epsilon > 0$. The ϵ-neighborhood of a is the set $(a - \epsilon, a + \epsilon)$.

$(\epsilon - K)$ Definition. x is the **limit** of (x_n) (or (x_n) converges to x) if for every $\epsilon > 0$, there exists $K = K(\epsilon) \in \mathbb{N}$ s.t. $|x_n - x| < \epsilon$ for all $n \geq K$.

Thm 3.1.1 If (x_n) converges, then it has exactly one limit.

3.2 Limit Theorems

NOTE: Unless specified otherwise, all limits in this section are defined for $n \to \infty$.

Definition. A sequence (x_n) is said to be **bounded** if there exists $M > 0$ s.t. $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Thm 3.2.1 Every convergent sequence is bounded.

The reverse (see Thm 3.3.1).

Thm 3.2.3 (Squeeze Thm) If $x_n \leq y_n \leq z_n$ for all $n \geq K_0$ and $\lim x_n = \lim z_n = a$, then $\lim y_n = a$.

Thm 3.2.4. If $|x_n| \to 0$, then $x_n \to 0$.

3.2.11 List of limits of some standard sequences.

- For a fixed number $|b| < 1$, we have $\lim b^n = 0$.
- For a fixed number $c > 0$, we have $\lim c^{\frac{1}{n}} = 1$.
- $\lim n^{\frac{1}{n}} = 1$
- $\lim \left(1 + \frac{1}{n}\right)^n = e$ → proven in Ex 3.3.5
- $n^k << n^l << a^n << b^n << n!$ if $k < l$ and $1 < a < b$.

- If (x_n) and (y_n) converges, taking the operations $+, -, \times, \div$ (only if $y_n \neq 0$ and $\lim y_n \neq 0$), $|\,|$, $\sqrt{\ }$ and the inequality signs $\leq$ and $\geq$ under limit preserves the convergence.

3.3 Monotone Sequences

Definition. We say that the sequence (x_n) is

1. increasing if $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$
2. decreasing if $x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots$
3. monotone if it is either increasing or decreasing.

Thm 3.3.1 (Monotone Convergence Theorem) If (x_n) is monotone and bounded, then (x_n) converges. In particular $\lim x_n$ equals to $\sup\{x_n : n \in \mathbb{N}\}$ if (x_n) is increasing, or $\inf\{x_n : n \in \mathbb{N}\}$ if (x_n) is decreasing.

3.4 Subsequences

Definition. Let (x_n) be a sequence and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence (x_{n_k}) is called a **subsequence** of (x_n)

Remark: If $(y_k) = (x_{n_k})$ is a subsequence of (x_n), then $n_k \geq k$.

Thm 3.4.1. If (x_n) converges to x, then any subsequence (x_{n_k}) also converges to x.

Corollary 3.4.2. If (x_n) has a divergent subsequence, then (x_n) diverges.
Corollary 3.4.3. If (x_n) has two convergent subsequences whose limits are not equal, then (x_n) diverges.

Thm 3.4.4 (Monotone Subsequence Theorem) Every sequence has a monotone subsequence

Thm 3.4.5 (Bolzano-Weierstrass Theorem) Every bounded sequence has a convergent subsequence.

3.5 Lim sup and lim inf

Definition. Let (x_n) be a sequence. $x \in \mathbb{R}$ is called the **subsequential limit** of (x_n) if (x_n) has a subsequence which converges to x.

Definition. Let $S(x_n)$ be the set of all subsequential limits of (x_n).

Definition. Let (x_n) be a **bounded** sequence. We know from Bolzano-Weierstrass Theorem, that $S(x_n)$ is non-empty and from Thm 3.2.11 that $S(x_n)$ is bounded.

1. $\limsup x_n := \sup S(x_n)$
2. $\liminf x_n := \inf S(x_n)$.

Thm 3.5.1 Let (x_n) be a bounded sequence and let $M = \limsup x_n$.

1. $\forall \epsilon > 0$, there are at most finitely many n's such that $x_n \geq M + \epsilon$. Equivalently, $\forall \epsilon > 0$, exists $K = K(\epsilon) \in \mathbb{N}$ s.t. $x_n < M + \epsilon$ for all $n \geq K$.
2. $\forall \epsilon > 0$, there are infinitely many n's such that $x_n > M - \epsilon$. Equivalently, $\forall \epsilon > 0$ exists a subsequence (x_{n_k}) of (x_n) s.t. $x_{n_k} > M - \epsilon$ for all $k \in \mathbb{N}$.

Thm 3.5.2 Let (x_n) be a bounded sequence and let $m = \liminf x_n$.

1. $\forall \epsilon > 0$, there are at most finitely many n's such that $x_n \leq m - \epsilon$. Equivalently, $\forall \epsilon > 0$, exists $K = K(\epsilon) \in \mathbb{N}$ s.t. $x_n > m - \epsilon$ for all $n \geq K$.
2. $\forall \epsilon > 0$, there are infinitely many n's such that $x_n < m + \epsilon$. Equivalently, $\forall \epsilon > 0$ exists a subsequence (x_{n_k}) of (x_n) s.t. $x_{n_k} < m + \epsilon$ for all $k \in \mathbb{N}$.

Thm 3.5.3 Let (x_n) be a bounded sequence. Then (x_n) converges (to x) iff $\limsup x_n = \liminf x_n = x$.

Thm 3.5.4. Let (x_n) and (y_n) be bounded sequences s.t. $x_n \leq y_n$ for all n. Then $\limsup x_n \leq \limsup y_n$ and $\liminf x_n \leq \liminf y_n$.

Thm 3.5.5 (Alternative definition of lim sup). Let (x_n) be a bounded sequence. Let $y_n = \sup\{x_k : k \geq n\}$, $\forall n \in \mathbb{N}$. Then the sequence (y_n) is decreasing and bounded, and $\limsup x_n = \lim y_n$.

3.6 Cauchy Criterion

Definition. A sequence (x_n) is called a **Cauchy sequence** if for every $\epsilon > 0$, there exists $K = K(\epsilon) \in \mathbb{N}$ s.t. $|x_n - x_m| < \epsilon$ for all $n > m \geq K$.

Thm 3.6.1 Every convergent sequence is Cauchy.

Thm 3.6.2 Every Cauchy sequence is bounded.

Thm 3.6.3 (Cauchy Criterion) Every Cauchy sequence is convergent.

A sequence (x_n) is called a contractive sequence if there exists some constant $0 < C < 1$, s.t. $|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$.

Thm 3.6.4 Every contractive sequence is Cauchy.

3.7 Properly divergent sequences

($M - K$) Definition. We say that a sequence (x_n) tends to ∞ if for every $M > 0$, there exists $K = K(M) \in \mathbb{N}$ s.t. $x_n > M$ for all $n \geq K$.

Similar definition (change signs) for tending to $-\infty$.

Thm 3.7.1 If a sequence (x_n) is increasing and not bounded above, then $x_n \to \infty$.

Definition. We call a sequence (x_n) **properly divergent** if either $x_n \to \infty$ or $x_n \to -\infty$.

(Bartle, Thm 3.6.3) A monotone sequence of real numbers is properly divergent iff it is unbounded.

(Bartle, Thm 3.6.4) Let (x_n) and (y_n) be 2 sequences of real numbers s.t. $x_n \leq y_n$ for all $n \in \mathbb{N}$. Then if $\lim x_n = +\infty$, then $\lim y_n = +\infty$. If $\lim y_n = -\infty$, then $\lim x_n = -\infty$.

(Bartle, Thm 3.6.5) Let (x_n) and (y_n) be 2 sequences of real numbers s.t. $\lim \frac{x_n}{y_n} = L$ for some $L \in \mathbb{R}, L > 0$. We have $\lim x_n = +\infty$ iff $\lim y_n = +\infty$.

Comparison of properly divergent sequences

Let (x_n) and (y_n) be two sequences of positive numbers s.t. $\lim x_n = \infty$ and $\lim y_n = \infty$. We write $x_n = o(y_n)$ or $x_n << y_n$ if $\lim \frac{x_n}{y_n} = 0$.

Remark 3.7.6. the above relation $<<$ is transitive.
Remark 3.7.7. $n^k << n^l << a^n << b^n << n!$ if $k < l$ and $1 < a < b$.

Indeterminate forms: $\frac{\infty}{\infty}, \frac{0}{0}, 0 \cdot \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$.

Tutorial results

- (Tut 3 Q2) If (a_n) converges and $|a_n + nb_n| < 1$ for all $n \in \mathbb{N}$, then the sequence (b_n) converges.
- (Tut 3 Q4) If (x_n) is convergent and (y_n) is divergent, then $(x_n + y_n)$ is divergent.
- (Tut 3 Q4) If (x_n) is divergent and (y_n) is convergent (not to 0), then $(x_n y_n)$ is divergent.
- (Tut 3 Q5) If $(x_n) \to x$, then AM, GM, HM converges to x.
- Example 3.3.5. $(e_n) = \left(1 + \frac{1}{n}\right)^n$ is increasing and bounded.
- (Tut 3 Q6. Similar to Ratio Test). Let (x_n) be a sequence of **positive** real numbers such that $L := \lim \frac{x_{n+1}}{x_n}$ exists. If $L < 1$, then (x_n) converges and $\lim x_n = 0$.
- (Tut 4 Q5) $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$. Similarly $\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n$
- (Tut 5 Q3) If $x_n > 0$ for all n, then $\lim x_n = 0$ iff $\lim \frac{1}{x_n} = \infty$.
- (Tut 5 Q4) Let (x_n) and (y_n) be sequence of *positive* numbers s.t. $\lim \frac{x_n}{y_n} = \infty$.
  - If $y_n \to \infty$, then $x_n \to \infty$.
  - If (x_n) is bounded, $y_n \to 0$.

4 Infinite Series

Notation: For the following chapter, $\sum$ is equivalent $\sum_{n=1}^{\infty}$ otherwise stated.

4.1 Definition and Examples

Definition 4.1.1. Given a series $\sum a_k$, define its n-th **partial sum** as $s_n = \sum a_k = a_1 + a_2 + \cdots + a_n$.

The sequence (s_n) is called the **sequence of partial sums** of the series $\sum a_k$.

Definition 4.1.2. Consider the sequence of partial sums (s_n) of the series $\sum a_k$. We say that the series **converges** (to S) and write $\sum a_k = \lim s_n = S$ if and only if (s_n) converges to a number $S \in \mathbb{R}$.

Remark 4.1.1. (Geometric Series). $\sum\limits_{k=1}^{\infty} ar^{k-1}$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges otherwise.

Thm 4.1.1 Let $\sum a_k$ and $\sum b_k$ be two convergent series and let $c \in \mathbb{R}$.

1. The series $\sum(a_k + b_k)$ is convergent and equals $\sum a_k$ + $\sum b_k$
2. The series $\sum ca_k$ is convergent and equals $c \sum a_k$

Thm 4.1.2 If $\sum a_n$ converges, then $\lim a_n = 0$.

Thm 4.1.3 (The n-th term divergence test).

1. If $\lim a_n \neq 0$ or do not exists, then $\sum a_n$ diverges.
2. If $\lim a_n = 0$, we can conclude NOTHING about the series $\sum a_n$

Thm 4.1.4 (Cauchy criterion for series). The series $\sum a_n$ converges iff for all $\epsilon > 0$, there exists $K = K(\epsilon) \in \mathbb{N}$ s.t. $|a_{n+1} + a_{n+2} + \cdots a_m| < \epsilon$ for all $m > n \geq K$.

4.2 Series with nonnegative terms

Goal: Given a series, test whether the series converges.

Definition 4.2.1. A series $\sum a_k$ is called an **eventually non-negative (positive) series** there exists $K \in \mathbb{N}$ s.t. $a_k \geq 0$ $(a_k > 0)$ for all $k \geq K$.

Thm 4.2.1 Let $\sum a_n$ be an eventually non-negative series. Then $\sum a_n$ converges iff the sequence (s_n) of partial sums is bounded above.

Remark 4.2.4 (p-series) If $p > 1$, then the p-**series** $\sum \frac{1}{n^p}$ converges. If $p \leq 1$, then the p-series diverges.

Thm 4.2.3 (Comparison Test) Consider 2 **eventually non-negative** series $\sum a_k$ and $\sum b_k$. Suppose there exists $K \in \mathbb{N}$ s.t. $0 \leq a_k \leq b_k$ for all $k \geq K$. If $\sum b_k$ converges, then $\sum a_k$ converges.

Remark 4.2.5. When applying comparison test, we try to compare a given series with either a **bigger but convergent** series or a **smaller but divergent** series. The two standard series often used in comparison tests are the p-series and the geometric series $\sum ar^{n-1}$

Thm 4.2.5 (Limit Comparison Test) Let 2 **eventually positive** series $\sum a_n$ and $\sum b_n$ and suppose that the limit $\rho = lim \frac{a_n}{b_n}$ exists.

1. If $\rho > 0$, then either both converge or both diverge
2. If $\rho = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

Intuition:

1. The two series resemble finite non-zero multiple of each other
2. $\sum a_n$ is smaller than $\sum b_n$ → similar to comparison test.

Thm 4.2.7 (Ratio Test). Let $\sum a_n$ be an **eventually positive** series. and suppose that the limit $\rho = \lim \frac{a_{n+1}}{a_n}$ exists.

- If $\rho < 1$, the series converges. ⟶ Note: can use $\limsup$ as well.
- If $\rho > 1$, the series diverges.
- No conclusion if $\rho = 1$.

Thm 4.2.8 (Root Test). Let $\sum a_n$ be an **eventually non-negative** series. and suppose that the $(a_n^{1/n})$ is a bounded sequence. Let $\rho = \limsup a_n^{1/n}$.

- If $\rho < 1$, the series converges.
- If $\rho > 1$, the series diverges.
- No conclusion if $\rho = 1$.

4.3 Alternating Series

Definition 4.3.1. An alternating series is a series of the form $\sum(-1)^{k+1}a_k$ or $\sum(-1)^k a_k$

Theorem 4.3.1 (Alternating Series Test) Let $\sum(-1)^k a_k$ be an alternating series. Suppose that $a_n \geq 0$ for all n, (a_n) is decreasing, and $\lim a_n = 0$. Then the series is convergent.

4.4 Absolute and Conditional convergence

For series with both +-ve and -ve terms.

Definition 4.4.1.

1. We say that the series $\sum a_n$ **converges absolutely** if the series $\sum |a_n|$ converges.
2. We say that the series $\sum a_n$ **converges conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Theorem 4.4.1 If the series $\sum a_n$ converges absolutely, then it converges.

Theorem 4.4.2 Every series is either absolutely convergent, conditionally convergent or divergent.

Dirichlet and Abel Tests

Abel's Lemma. Let $X := (x_n)$ and $Y := (y_n)$ be sequences in $\mathbb{R}$ and let the partial sums of $\sum y_n$ be denoted by (s_n) with $s_0 = 0$. If $m > n$, then
$\sum_{k=n+1}^{m} x_k y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k$

Dirichlet's Test. If (x_n) is a decreasing sequence with $\lim x_n = 0$ and if the partial sums (s_n) of $\sum y_n$ are bounded, then the series $\sum x_n y_n$ is convergent.

Abel's Test. If (x_n) is a convergent monotone sequence and the series $\sum y_n$ is convergent, then the series $\sum x_n y_n$ is also convergent.

Tutorial Results

- (Tut 6 Q2). If $\sum a_n$ is convergent, the series of its AM is divergent.

5 Limits of Functions

Definition 5.2.1 Let $\emptyset \neq A \subseteq \mathbb{R}$. A real number c is a **cluster point** of A if for every $\delta > 0$, there exists a point $x \in A \backslash \{c\}$ s.t. $0 < |x - c| < \delta$.

Remark. A cluster point c of a set A need not be an element of A.

Proposition 5.2.1. A real number c is a cluster point of $\emptyset \neq A \subseteq \mathbb{R}$ if and only if there exists a sequence (a_n) in $A \backslash \{c\}$ s.t. $\lim a_n = c$.

Definition 5.2.2. ($\epsilon - \delta$ definition of limit). Let $f : A \to \mathbb{R}$ be a function, where $\emptyset \neq A \subseteq \mathbb{R}$ and let c be a cluster point of A. We say that $\lim_{x \to c} f(x) = L$ if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ s.t. $|f(x) - L| < \epsilon$ for all $x \in A$ satisfying $0 < |x - c| < \delta$.

If the limit of f at $x = c$ does not exist (in $\mathbb{R}$), we say that f **diverges** at $x = c$.

Definition:

- If $h > 0$, then the h**-neighborhood** of the point a is the set $V_h(a) = \{x : |x - a| < h\} = (a - h, a + h)$.
- Define $V_h^*(a) = V_h(a) \backslash \{a\} = \{x : 0 < |x - a| < h\}$ as the **deleted neighborhood** of a.

Remark:

1. If c is not a cluster point of A, then there exists $\delta > 0$ s.t. $A \cap V_\delta^*(c) = \emptyset$.
2. We can rewrite definition 5.2.2 as $\lim_{x \to c} f(x) = L$ iff for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ s.t. $f(A \cap V_\delta^*(c)) \subseteq V_\epsilon(L)$.
3. The limit of a function at a cluster point is unique if the limit exists.

Theorem 5.2.3. (Sequential Criterion for Limits). Let $\emptyset \neq A \subseteq \mathbb{R}$ and let c be a cluster point of A. Suppose that $f : A \to \mathbb{R}$ and $L \in \mathbb{R}$. Then the following statements are equivalent:

1. $\lim_{x \to c} f(x) = L$
2. For every sequence (x_n) in $A \backslash \{c\}$ s.t. $\lim x_n = c$, one has $\lim f(x_n) = L$

Theorem 5.2.4 (Uniqueness of limits)
If $f : A \to \mathbb{R}$ is a function and c is a cluster point of A, the the limit of f at $x = c$ is unique if it exists.

Remark 5.2.6 (Divergence Criteria). Let $f : A \to \mathbb{R}$ be a function and let c be a cluster point of A. To prove $\lim_{x \to c} f(x)$ does not exist ,either:

1. Find a sequence (x_n) in $A \backslash \{c\}$ s.t. $\lim x_n = c$, but the sequence $(f(x_n))$ diverges.
2. Find 2 sequences (x_n) and (y_n) in $A \backslash \{c\}$ s.t. $\lim x_n = c = \lim y_n$ but $\lim f(x_n) \neq \lim f(y_n)$.

5.3 Limit Theorems

Assume $f : A \to \mathbb{R}$ is a function and c is a cluster point of A.

Theorem 5.3.1 If $\lim_{x \to c} f(x)$ exists, then there exist constants $M, \delta > 0$ s.t. $|f(x)| \leq M$ for all $x \in A$ satisfying $0 < |x - c| < \delta$.

Basic Principle 5.3.5 Suppose there exists a deleted neighborhood $V_h^*(c)$ (with $h > 0$) s.t. $f(x) = g(x)$ for all $x \in A \cap V_h^*(c)$, then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$ provided one of these limits exist.

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exists, applying the operations $+, -, \times, \div$ (only if $g(x) \neq 0$ for all $x \in A$), $|\,|$, $\sqrt{}$ and the inequality signs $\leq$ and $\geq$ between the functions is the same as applying it to their limits.

Theorem 5.3.7. (Squeeze Theorem) Let f, g, h be three functions and let c be a cluster point of A. Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in A$ satisfying $0 < |x - c| < \delta$ and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$.

Theorem 5.3.8 If $\lim_{x \to c} f(x) = L$ exists and $L > 0$, then exists $\delta > 0$ s.t. $f(x) > 0$ for all $x \in A$ satisfying $0 < |x - c| < \delta$.

5.4 One sided limits

Definition 5.4.1.

1. Let c be a cluster point of $A \cap (c, \infty)$. We say that L is the **right-hand limit** of f at c, denoted by $\lim_{x \to c^+} f(x)$, if for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ s.t.

$|f(x) - L| < \epsilon$ for all $x \in A$ satisfying $c < x < c + \delta$.

2. Let c be a cluster point of $A \cap (-\infty, c)$. We say that L is the **left-hand limit** of f at c, denoted by $\lim_{x \to c^-} f(x)$, if for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ s.t. $|f(x) - L| < \epsilon$ for all $x \in A$ satisfying $c - \delta < x < c$.

Theorem 5.4.2 $\lim_{x \to c} f(x) = L$ iff both left-hand limit and right-hand limit exists and is equal to L.

Theorem 5.4.4. (Sequential Criterion for right-hand Limits).Suppose that $f : A \to \mathbb{R}$ and let c be a cluster point of $A \cap (c, \infty)$. Then the following statements are equivalent:

1. $\lim_{x \to c^+} f(x) = L$
2. For every sequence (x_n) in $A \cap (c, \infty)$ s.t. $\lim x_n = c$, one has $\lim f(x_n) = L$

There is a similar sequential criterion for the left-hand limit.

Note: All limit theorems (squeeze, operations, basic principle 5.3.5) also holds for one-sided limits.

5.5 Infinite Limits

Definition: We say that $\lim_{x \to c} f(x) = +\infty$ if for every $M > 0$, there exists $\delta = \delta(M) > 0$ s.t $f(x) > M$ for all $x \in A$ satisfying $0 < |x - c| < \delta$.

We have a similar definition for tending to $-\infty$.

Note

- There is a similar sequential criterion for the infinite limits, (Theorem 5.2.3), and infinite one-sided limits (Theorem 5.4.4) with $L = \infty$
- We have a stronger statement than Squeeze Theorem, i.e. if $f(x) \geq g(x)$ for all x in domain and $\lim g(x) = \infty$, then $\lim f(x) = \infty$.

5.6 Limits at infinity

Definition 5.6.1.

1. Suppose A is not bounded above. We say that $\lim_{x \to \infty} f(x) = L$ if for every $\epsilon > 0$, there exists $M = M(\epsilon) > 0$ s.t. $|f(x) - L| < \epsilon$ for all $x \in A$ and $x > M$.
2. Suppose A is not bounded below. We say that $\lim_{x \to -\infty} f(x) = L$ if for every $\epsilon > 0$, there exists $M = M(\epsilon) < 0$ s.t. $|f(x) - L| < \epsilon$ for all $x \in A$ and $x < M$.

Remark 5.6.2. The concept of the limit of a sequence is a special case of the above definition (with $A = \mathbb{N}$).

Theorem 5.6.3 (Sequential Criterion for limit at infinity) Suppose that $f : A \to \mathbb{R}$ and suppose A is not bounded above. Then the following statements are equivalent:

1. $\lim\limits_{x \to \infty} f(x) = L$
2. For every sequence (x_n) in A s.t. $\lim x_n = \infty$, one has $\lim f(x_n) = L$

Note: All limit theorems (squeeze, operations, basic principle 5.3.5) also holds for limits at infinity.

5.7 Infinite limits at infinity

Definition 5.7.1. Suppose A is not bounded above. We say that $\lim\limits_{x \to \infty} f(x) = \infty$ if for every $M > 0$, there exists $K = K(M) > 0$ s.t. $f(x) > M$ for all $x \in A$ and $x > K$.

Note

- There is a similar sequential criterion (Theorem 5.6.3 with $L = \infty$)

Tutorial Results

- (Tut 8 Q3) Suppose $\lim\limits_{x \to a} f(x) = L > 0$ and $\lim\limits_{x \to a} g(x) = \infty$, then $\lim\limits_{x \to a} f(x)g(x) = \infty$.

6 Continuous Functions

Definition 6.1.1. ($\epsilon - \delta$ definition of continuity). A function $f : A \to \mathbb{R}$ is said to be **continuous at** $x = a \in A$ if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon, a) > 0$ s.t. $|f(x) - f(a)| < \epsilon$ for all $x \in A$ satisfying $|x - a| < \delta$.

- If f is not continuous at a, we say that f is **discontinuous** at $x = a$ and a is a **point of discontinuity** of f.
- If f is continuous at every point in A, we say that f is continuous on A.
- If $a \in A$ is not a cluster point of A, then f is always continuous at a since there exists $\delta > 0$ s.t. $A \cap (a - \delta, a + \delta) = \{a\}$
- If the choice of δ only depends on ϵ, the function is said to be uniformly continuous

Theorem 6.1.1 (Continuity in terms of limits). If $a \in A$ is a cluster point of A, then f is continuous at $x = a$ iff $\lim\limits_{x \to a} f(x) = f(a)$.

Theorem 6.1.6. (Sequential Criterion for Continuity). We have something similar to Theorem 5.2.3 (Sequential criterion for limits) with $L = f(a)$ and domain of (x_n) is A.

Summary of common continuous functions

1. Polynomials, absolute-value functions, sine and cosine functions are continuous on $\mathbb{R}$.
2. nth-root functions are continuous on $(0, \infty)$.

3. Rational functions $p(x)/q(x)$ are continuous everywhere except at the zeros of q.
4. Floor function is continuous on $\mathbb{R}\backslash\mathbb{Z}$ (and is discontinuous at each $a \in \mathbb{Z}$)

Remark: Sometimes we can "save" a function which is discontinuous at a point, i.e. if $\lim_{x \to a} f(x) = L$ exists but $f(a)$ is not defined, then we can simply define $f(a) = L$ and the resulting function will now be continuous at a.

6.2 Combinations of continuous functions

Theorem 6.2.1. Let $A \subseteq \mathbb{R}$ and let $f, g : A \to \mathbb{R}$ be continuous functions at $a \in A$. Let $k \in \mathbb{R}$. Then the functions $f + g, f - g, kf$ and $f \cdot g$ are all continuous at $x = a$. If $g(a) \neq 0$, then the function f/g is also continuous at $x = a$.

Theorem 6.2.2. Composition of continuous functions is continuous provided the image of the inner functions fall inside the domain of the outer function.

6.3 Continuous functions on intervals

Definition 6.3.1. A function $f : A \to \mathbb{R}$ is said to be **bounded on A** if the image $f(A)$ is a bounded set.

Theorem 6.3.1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the **closed bounded** interval $[a, b]$. Then f is bounded on $[a, b]$.

Definition 6.3.2. We say that f has an **absolute maximum** on A, denoted $\max f(A)$, if there exists $y \in A$ s.t. $f(y) \geq f(x)$ for all $x \in A$.

Theorem 6.3.3. (Extreme Value Theorem) Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function on the closed bounded interval $[a, b]$. Then f has an absolute maximum and an absolute minimum on $[a, b]$.

Theorem 6.3.5. (Intermediate Value Theorem) Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function on the closed bounded interval $[a, b]$. Then for any number L strictly between $f(a)$ and $f(b)$, there exists $c \in (a, b)$ s.t. $f(c) = L$.

Corollary: We have $[f(a), f(b)] \subseteq f([a, b])$

Theorem 6.3.7. A continuous function sends a closed bounded interval onto a closed bounded interval.

Theorem 6.3.10. (Preservation of Intervals) Let I be an interval in $\mathbb{R}$ and suppose a function $f : I \to \mathbb{R}$ is continuous on I. Then $f(I)$ is an interval.

6.4 Monotone and Inverse functions on intervals

Definition 6.4.1.

1. f is said to be **increasing on A** if $x_1, x_2 \in A$ and $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$. It is strictly increasing if $f(x_1) < f(x_2)$
2. Similar definition for decreasing.
3. A function is **(strictly) monotone** if it is either (strictly) increasing or (strictly) decreasing on A.

Theorem 6.4.2. A monotone function defined on an interval always has one-sided limits. Let $f : I \to \mathbb{R}$ be increasing on I. If $c \in I$ is not an endpoint of I, then

1. $\lim_{x \to c^-} f(x) = \sup\{f(x) : x \in I, x < c\}$
2. $\lim_{x \to c^+} f(x) = \inf\{f(x) : x \in I, x > c\}$
3. $\lim_{x \to c^-} f(x) \leq f(c) \leq \lim_{x \to c^+} f(x)$

Remark: If f is increasing and discontinuous at c, then $\lim_{x \to c^-} f(x) < \lim_{x \to c^+} f(x)$ and the difference is called the **jump** of f at c.

Theorem 6.4.6. (Continuous Inverse Theorem) Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a strictly monotone function. If f is continuous on I, then its inverse function $f^{-1} : f(I) \to \mathbb{R}$ is strictly monotone and continuous on $f(I)$.

Example: the n-th root function is continuous and strictly increasing on $\mathbb{R}$.

6.5 Uniform continuity

Definition 6.5.1. Let $\emptyset \subsetneq A \subseteq \mathbb{R}$. A function $f : A \to \mathbb{R}$ is said to be **uniformly continuous on** A if for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ s.t. $|f(x) - f(u)| < \epsilon$ for any $x, u \in A$ satisfying $|x - u| < \delta$.

Remark:

1. uniformly continuous on A $\longrightarrow$ continuous on A.
2. uniformly continuous on A $\longrightarrow$ uniformly continuous on any nonempty subset B of A.

Theorem 6.5.3. (Sequential criterion for uniform continuity) Let $\emptyset \neq A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. Then the following statements are equivalent:

1. f is uniformly continuous on A.
2. For any two sequences (x_n) and (u_n) in A s.t. $x_n - u_n \to 0$, one has $f(x_n) - f(u_n) \to 0$.

Theorem 6.5.5. (Heine-Cantor Theorem) Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function on the closed bounded interval $[a, b]$. Then f is uniformly continuous on $[a, b]$.

Lipschitz condition. Let A be a nonempty subset of $\mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function satisfying the Lipschitz condition on A: there exists a constant $K > 0$ s.t.

$|f(x) - f(y)| \le K|x - y|$ for all $x, y \in A$. Then f is uniformly continuous on A.

Theorem. If $f : A \to \mathbb{R}$ is uniformly continuous on a subset $A \subseteq \mathbb{R}$ and if (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

Continuous Extension Theorem. A function f is uniformly continuous on the interval (a, b) iff it can be defined at the endpoints a and b s.t. the extended function is continuous on $[a, b]$.

Tutorial Results

- (Tut 8 Q8) If $0 < C < 1$ and $|f(x) - f(y)| \le C|x - y|$ then there exist a unique point a s.t. $f(a) = a$.
- (Tut 9 Q7) Suppose $f : [a, b] \to \mathbb{R}$ is continuous and injective on $[a, b]$. Then f is strictly monotone.
- (Tut 10 Q4) If f is uniformly continuous on an interval I and there is a positive number k s.t. $|f(x)| \ge k$ for all $x \in I$, then $1/f(x)$ is uniformly continuous on I.
- (20/21 Sem 1 Q5) If f is a continuous function. Then f is strictly monotone iff f is injective.

7 Metric Spaces

7.1 Metric Space

Definition 7.1.1. Let $S \neq \emptyset$. A **metric** on the set S is a function $d : S \times S \to \mathbb{R}$ that satisfies

1. (*Positivity*) $d(x, y) \ge 0$ for all $x, y \in S$
2. (*Definiteness*) $d(x, y) = 0$ iff $x = y$
3. (*Symmetry*) $d(x, y) = d(y, x)$ for all $x, y \in S$
4. (*Triangle inequality*) $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in S$.

A **metric space** (S, d) is a set S together with a metric d on S. The metric d is also called a **distance function** on S.

Remark 7.1.5. Define the **Euclidean distance function** on $\mathbb{R}^n$ as

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then $(\mathbb{R}^n, d)$ forms the n**-dimensional Eucliean space**.

Let (S, d) be a metric space, and let $A \subseteq S$. The **induced metric** $d_A : A \times A \to \mathbb{R}$ on A is defined as $d_A(x, y) := d(x, y)$ for all $x, y \in A$. Then (A, d_A) is called a **metric subspace** of (X, d).

Other metrics on $\mathbb{R}^n$

1. $d_1(x,y) := \sum_{i=1}^{n} |x_i - y_i|$
2. $d_\infty(x,y) := \max_{1\le i\le n} |x_i - y_i|$. → (actually it is supremum)
3. (discrete metric) $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ otherwise.

Remark: In view of the other metrics in $\mathbb{R}^n$, then we often denote the Euclidean metric as d_2.

Exercise 7.1.10. We have $d_\infty(x,y) \le d_2(x,y) \le d_1(x,y) \le n \cdot d_\infty(x,y)$.

7.2 Neighborhood, Convergence

Definition 7.2.1. For $\epsilon > 0$, then the **ϵ-neighborhood** of the point $c \in S$ is the set $V_\epsilon(c) = \{x \in S : d(x,c) < \epsilon\}$.

Definition 7.2.3. A set U is called a **neighborhood** of x if U contains an ϵ-neighborhood of x for some $\epsilon > 0$.

Definition 7.2.6. Let (x_n) be a sequence of points in (S,d). Let $x \in S$. The sequence (x_n) is said to **converge to x in S** (with respect to d) if for every $\epsilon > 0$, exists $K = K(\epsilon) \in \mathbb{N}$ s.t. $x_n \in V_\epsilon(x)$ for all $n \ge K$

Remark: $\lim_{n\to\infty} x_n = x$ iff $\lim_{n\to\infty} d(x_n, x) = 0$

7.3 Open Sets, Closed Sets

Let (S,d) be a metric space.

Definition 7.3.1. A subset G of S is said to be an **open** set in S if for each $x \in G$, there exists a neighborhood V of x s.t. $V \subseteq G$.

Comment: Roughly speaking, an open set is a set whose "boundary points" are all excluded from the set.

Example: Let $a, b \in \mathbb{R}$ s.t. $a < b$. Then the open interval (a,b) is open.

Theorem 7.3.5. Let $a \in S$ and $r > 0$. Then the r-neighborhood $V_r(a)$ of a is open in S.

Theorem 7.3.7. (Open Set Properties) Let (S,d) be a metric space

1. The empty set $\emptyset$ and S are open.
2. Let $\{G_\lambda : \lambda \in \Lambda\}$ be a collection of open subsets of S, i.e. G_λ is open for each $\lambda \in \Lambda$. Then $\bigcup_{\lambda\in\Lambda} G_\lambda$ is open.
3. Let $G_1, G_2, \ldots, G_n$ be n (finite) open subsets of S. Then $\bigcap_{k=1}^{n} G_k$ is open.

Definition 7.3.10. A subset F of S is said to be an **closed** set in S if the complement $C(F) := S \backslash F$ is open in S.

Remark: G is open in S iff $S \backslash G$ is closed in S.

Example:

- The set $\bar{V}_r(a) := \{x \in S : d(x,a) \leq r\}$ is closed in S.
- Let $a, b \in \mathbb{R}$ s.t. $a < b$. Then the closed interval $[a,b]$ is closed.

Theorem 7.3.15. (Closed Set Properties) Let (S,d) be a metric space

1. The empty set $\emptyset$ and S are closed.
2. Let $\{F_\lambda : \lambda \in \Lambda\}$ be a collection of closed subsets of S, i.e. F_λ is closed for each $\lambda \in \Lambda$. Then $\bigcap_{\lambda \in \Lambda} F_\lambda$ is closed.
3. Let $F_1, F_2, \ldots, F_n$ be n (finite) closed subsets of S. Then $\bigcup_{k=1}^{n} F_k$ is closed.

Theorem 7.3.18. (Characterization of Closed Sets) Let $F \subseteq S$. The following statements are equivalent:

1. F is closed in S.
2. Every convergent sequence $(x_n) \subseteq F$ has its limit in F, i.e. one has $\lim_{n \to \infty} x_n \in F$.

Some additional definition.

- $\mathbb{Q}$ is neither open nor closed.
- A point $x \in \mathbb{R}$ is said to be an **interior point** of $A \subseteq \mathbb{R}$ if there is a neighborhood V of x s.t. $V \subseteq A$.
- A set A is open iff every point of A is an interior point of A.
- A point $x \in \mathbb{R}$ is said to be an **boundary point** of $A \subseteq \mathbb{R}$ if everyneighborhood V of x contains points in A and points in $C(A)$.
- A set A is open iff it does not contain any of its boundary points.
- A set A is closed iff it contains all its boundary points.

7.4 Continuity in terms of open sets

Context: Let (S_1, d_1) and (S_2, d_2) be metric spaces, and let $A \subseteq S_1$. Let $f : A \to S_2$ be a function.

Definition 7.4.1. The function f is said to be **continuous** at a point $c \in A$ if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon, c) > 0$ s.t. $d_2(f(x), f(c)) < \epsilon$ for all $x \in A$ satisfying $d_1(x,c) < \delta$.

Or equivalently: $f(A \cap V_\delta(c)) \subseteq V_\epsilon(f(c))$

Definition 7.4.2. Let $f : A \to B$ be a function and let $G \subseteq B$. Then the **inverse image** of G under f is given by $f^{-1}(G) := \{x \in A : f(x) \in G\} \subseteq A$.

Remark: We have $f(f^{-1}(G)) \subseteq G$.

Theorem 7.4.3. (Global Continuity Theorem) The following statements are equivalent:

1. f is continuous on A.
2. For every *open* set $G \subseteq S_2$, there exists an *open* set $H \subseteq S_1$ s.t. $f^{-1}(G) = A \cap H$

Corollary 7.4.5. A function $f : S_1 \to S_2$ is continuous on S_1 iff the inverse image $f^{-1}(G)$ is open in S_1 for every open set G in S_2.

Remark: The above corollary also works if I change the word "open" to "closed".

Theorem 7.4.8. (Sequential Criterion for Continuity) The following statements are equivalent:

1. f is continuous at c.
2. For every sequence (x_n) in A s.t. $x_n \to c$, one has $f(x_n) \to f(c)$.

7.5. Sequential compactness

Let (S, d) be a metric space.

Definition 7.5.1. A subset $A \subseteq S$ is said to be **bounded** if there exists $x_0 \in S$ and $M > 0$ s.t. $d(x, x_0) \leq M$ for all $x \in A$.

Definition 7.5.4. A subset $A \subseteq S$ is said to be **sequentially compact** if every sequence in A has a convergent subsequence whose limit is in A.

Theorem 7.5.6. Suppose a subset $A \subseteq S$ is sequentially compact, then A is closed and bounded in S.

Theorem 7.5.9. (Heine-Borel Theorem) Let $k \in \mathbb{N}$. Consider the Euclidean k-space $(\mathbb{R}^k, d_2)$ where d_2 is the Euclidean metric on $\mathbb{R}^k$. Then a subset $A \subseteq \mathbb{R}^k$ is sequentially compact iff A is closed and bounded in $(\mathbb{R}^k, d_2)$

Remark: Generalized version of Heine-Cantor Theorem by substituting "closed and bounded" with "compact".

Theorem 7.5.10. Continuous functions preserve sequentially compact sets.

Theorem 7.5.11. (Extreme Value Theorem) Let (S, d) be a metric space and let $\emptyset \neq A \subseteq S$ be a **sequentially compact** set. Suppose $f : A \to \mathbb{R}$ be a **continuous** (real-valued) function on A. Then there exist $x_1, x_2 \in A$ s.t. $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in A$.

Corollary 7.5.12. EVT can be generalized to higher dimensions $(\mathbb{R}^k, d_2)$.

7.6 Compactness

Definition 7.6.1. Let (S, d) be a metric space and let $A \subseteq S$.

1. An **open cover** of A is a collection T of open subsets of S s.t. $\bigcup_{G \in T} G \supseteq A$.
2. An open cover T of A is said to have a **finite subcover** if there exist finitely many open sets $G_1, G_2, \dots, G_n \in T$ s.t. $G_1 \cup G_2 \cup \dots \cup G_n \supseteq A$.

Definition 7.6.3. A subset $A \subseteq S$ is said to be **compact** in (S, d) if every open cover of A has a finite subcover.

Example:

- Every finite subset of $\mathbb{R}$ is compact
- $[0, \infty)$ and $(0, 1)$ is not compact.

Theorem 7.6.6. A is compact iff A is sequentially compact.

Extending Continuous Inverse Theorem, we have
Theorem. If K is a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is injective and continuous, then f^{-1} is continuous on $f(K)$.

Some results

- If G is an open set and F is a closed set, then $G \backslash F$ is an open set and $F \backslash G$ is a closed set.
- If F is a closed subset of a compact set K in $\mathbb{R}$, then F is compact.
- If K_1 and K_2 are compact sets, then $K_1 \cup K_2$ and $K_1 \cap K_2$ is compact.
- Let $K \neq \emptyset$ be a compact set, then $\inf K$ and $\sup K$ exists and belong to K.
- If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the set $\{x \in \mathbb{R} : f(x) < \alpha\}$ is open; the set $\{x \in \mathbb{R} : f(x) \leq \alpha\}$ and the set $\{x \in \mathbb{R} : f(x) = k\}$ is closed.

Appendix

2.6 Finite and Infinite sets

Definition 2.6.0. Let $f : A \to B$ be a function. Then,

1. f is **injective** if for all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
2. f is **surjective** if $f(A) = B$, i.e. for every $y \in B$, there exists $x \in A$ s.t. $f(x) = y$.
3. f is **bijective** if f is both injective and surjective.

Definition 2.6.1.

1. A set is **finite**, if it have $n \geq 0$ elements. A set S have $n > 0$ elements iff there is a bijection from S onto the set $\{1, 2, \dots, n\}$ (or from the set $\{1, 2, \dots, n\}$ to S).

2. A set S is said to be **denumerable** (or **countably infinite**) if there exists a surjection of $\mathbb{N}$ onto S (or an injection from S onto $\mathbb{N}$).
3. S is countable if it is finite or countably infinite.
4. A set is **infinite** if it is not finite. A set is **uncountable** if it is not countable.

Uniqueness Thm: If S is a finite set, then the numbers of elements in S is a unique number in $\mathbb{N}$. Moreover, the set $\mathbb{N}$ is an infinite set.

Lemma 2.6.4. Any subset A of $\mathbb{N}$ is countable.

The set $\mathbb{N} \times \mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are denumerable. The interval $I = [0, 1]$ is uncountable.

Prop 2.6.5. Let $A \subseteq B$

1. If B is finite, then A is finite.
2. If B is countable, then A is countable.

Prop 2.6.6. If A_m is a countable set for each $m \in \mathbb{N}$, then the union $A := \bigcup_{m=1}^{\infty} A_m$ is countable.

Cantor's Thm: If A is any set, then there is no surjection of A onto the set $\pi(A)$ of all subsets of A

4.6 Rearrangements of series

Definition. A series $\sum b_n$ is called a **rearrangement** of a series $\sum a_n$ if there is a bijection $f : \mathbb{N} \to \mathbb{N}$ s.t. $b_n = a_{f(n)}$ for all $n \in \mathbb{N}$.

Theorem 4.6.2 (Rearrangement Theorem). Let $\sum a_k$ be an **absolutely convergent** series. Then, any rearragement $\sum b_k$ of $\sum a_k$ also converges and we have $\sum b_k = \sum a_k$.

Comment: Riemann showed that a conditionally convergent series can be rearranged s.t. $\sum a_n = c$ for any arbitrary constant c

4.7 Why is e irrational?

Theorem 4.7.1

1. $e = \sum \frac{1}{n!}$ → Proof: Using $\lim \left(1 + \frac{1}{n}\right)^n = e$
2. For each $n \in \mathbb{N}$, $e - \sum\limits_{j=0}^{n} \frac{1}{j!} \leq \frac{1}{n(n!)}$

Theorem 4.7.2 The Euler number e is irrational.

6.6 Applications of the notion "uniform continuity" to approximate continuous functions

Definition 6.6.1. Let $I \subseteq \mathbb{R}$ be an interval. Then a function $s : I \to \mathbb{R}$ is said to be a **step function** if I can be partitioned into a union of finite number of subintervals s.t. s

restricts to a constant function on each of these subintervals.

Theorem 6.6.2. Let $f:[a,b]\to\mathbb{R}$ be continuous on the closed bounded interval $[a,b]$. Then for any given $\epsilon>0$, there exists a step function $s_\epsilon:[a,b]\to\mathbb{R}$ s.t. $|f(x)-s_\epsilon(x)|<\epsilon$ for all $x\in[a,b]$.

Definition 6.6.3. A function $g:[a,b]\to\mathbb{R}$ is **piecewise linear on** $[a,b]$ if the interval $[a,b]$ can be partitioned into a finite number of subintervals s.t. the restriction of g to each subinterval is a linear function on the subinterval.

Theorem 6.6.4. Let $f:[a,b]\to\mathbb{R}$ be continuous on the closed bounded interval $[a,b]$. Then for any given $\epsilon>0$, there exists a continuous piecewise linear function $g_\epsilon:[a,b]\to\mathbb{R}$ s.t. $|f(x)-g_\epsilon(x)|<\epsilon$ for all $x\in[a,b]$.

7.7 Connectedness

Definition 7.7.1. Let (S,d) be a metric space.

1. A subset A of S is **disconnected** if there exist open subsets G,H of S s.t. $G\cap A\neq\emptyset$, $H\cap A\neq\emptyset$, $G\cap H\cap A=\emptyset$ and $A\subseteq G\cup H$.
2. A subset A of S is **connected** if A is not disconnected.

Theorem 7.7.6. Consider the metric space $(\mathbb{R},d)$ where d is the usual metric on $\mathbb{R}$. A subset A of $\mathbb{R}$ is connected iff A is an interval.

Theorem 7.7.7. Continuous functions preserve connected sets.

Theorem 7.7.9. (Intermediate Value Theorem). Let (S,d) be a metric space, and let $A\neq\emptyset$ be a connected subset of S. Suppose a function $f:A\to\mathbb{R}$ is continuous on A. If $a,b\in A$ and $f(a)<L<f(b)$, then there exists $c\in A$ s.t. $f(c)=L$.