$$\sum_{i=1}^{\infty} iy^i = \frac{y}{(1-y)^2}$$

$$\sum_{i=1}^{n} iy^i = \frac{y(1-y^n) - ny^{n+1}(1-y)}{(1-y)^2}$$

# 1 Theory of Interest

## 1.1 Interest

Accumulation function $a(t)$ is the accumulated value of 1 dollar at time $t \geq 0$. We have $a(0) = 1$
.

Let $r$ be the annual rate of interest

1. Simple Interest: $a(t) = 1 + rt$
2. Compound Interest: $a(t) = (1+r)^t$

**Nominal** rate convertible $p$ times a year is denoted by $r^{(p)}$. The interest to be paid over 1 period is $\frac{r^{(p)}}{p}$. $p$ is called the **frequency of compounding**

Two nominal interest rates $r^{(p)}$ and $r^{(q)}$ are **equivalent** if $\left(1 + \frac{r^{(p)}}{p}\right)^p = \left(1 + \frac{r^{(q)}}{q}\right)^q$

The **effective** annual interest rate $r_e = \left(1 + \frac{r^{(p)}}{p}\right)^p - 1$

Interest is **compounded continuously** when $p \to \infty$. Then $a(1) = \lim_{p\to\infty} \left(1 + \frac{r^{(p)}}{p}\right)^p = e^{r^{(\infty)}}$. $r^{(\infty)}$ is known as *continuously compounded rate of interest*

**Force of interest** from accumulation function is defined as $\delta(t) = \frac{a'(t)}{a(t)}$

Thus, we have $a(p,q) = \exp\left(\int_p^q \delta(t)dt\right)$

## 1.2 Present Value and Time Value

For a cashflow $C = \{(c_1,t_1),(c_2,t_2),\ldots(c_n,t_n)\}$ consisting of a series of payments $c_i$ received at time $t_i$, the **present value** (PV) of this cashflow is $PV(C) = \sum_{i=1}^{n} \frac{c_i}{a(t_i)}$

For the special case when $t_i = i - 1$, we can rewrite the cashflow above as $C = (c_1, c_2, \ldots, c_n)$

Time value at time $t$ ($TV_t$) of a cashflow $C$ is $PV_{t=0}(C) \times a(t)$

(Principle of Equivalence). 2 cashflow streams are **equivalent** iff they have the same time value at any point in time.

(Equation of Value) $PV(C) = 0$.

Any non-negative root $r$ of the equation of value is called the **yield** or **IRR** of the cashflow stream $C$.

(Newton Raphson Method)

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$$

## 1.3 Annuities

An annuity is a series of payments made at regular intervals.

1. An **annuity-due** is one for which payments are made at the beginning of each period
2. An **annuity immediate** is one for which payments are made at the end of each period.
3. Perpetual annuity (perpetuity) is an annuity with an infinite number of payments.

Note: beginning of $n+1$-th year = end of $n$-th year → $t = n$

## 1.4 Loans

If a loan $L$ made at $t = 0$ is to be repaid with the cashflow stream $C = \{(c_1, t_1), (c_2, t_2), \ldots (c_n, t_n)\}$, then $L = PV(C)$

The loan balance immediately after the $m$-th installment has been paid is the TV at $t = m$ of the remaining $(n - m)$ installment payments.

# 2 Bonds and Term Structure

Terminology

1. Face value (par value): the amount based on which periodic interest payments are computed
2. Redemption value (maturity value): the amount to be repaid at the end of loan
3. Maturity date (redemption date): the date on which the loan will be fully repaid
4. Coupon rate - the bond's interest payments as a percentage of the face value to be made at regular intervals during the term of the loan.

## 2.2 Bond Valuations

Let $P$ be the current price, $F$ the face value, $R$ be the redemption value, $c$ the nominal coupon rate, $m$ the number of coupon payments per year, $n$ total number of coupon payments and $\lambda$ the nominal yield.

$P = PV(C)$ where $C$ is made up of coupon payments of $cF/m$ at time $t = 1/m, 2/m. \ldots, n/m$ and a redemption of $R$ at $t = n/m$

A bond is priced at a *premium* if $P > F$, at *par* if $P = F$ and at a *discount* if $P < F$

$P = \frac{R}{(1+\lambda/m)^n} + \frac{cF}{\lambda}\left[1 - \frac{1}{(1+\lambda/m)^n}\right].$

If $F = R$, the above can be written as $P = F + F\left(\frac{c-\lambda}{\lambda}\right)\left[1 - \frac{1}{(1+\lambda/m)^n}\right]$. Hence a bond is priced at a premium if $c > \lambda$, at par if $c = \lambda$ and at a discount if $c < \lambda$

If $K = \frac{F}{(1+\lambda/m)^n}$, we can write $P = \frac{R}{F}K + \frac{c}{\lambda}(F - K)$

A **perpetual** bond (or consol) is a bond that never matures, i.e. $n \to \infty$. We have $P = \frac{cF}{\lambda}$

Bond Price between Coupon Payments at the same Bond Yield: $P_{k+\epsilon} = (1 + \lambda/m)^{\epsilon} P_k$

## 2.3 Duration

The **Macaulay Duration** $D$ is the weighted average time to maturity $t_i$ of the cashflow stream $C = \{(c_i, t_i) \mid i \in [1, n]\}$ with weights given by $w_i = \frac{PV(c_i)}{\sum PV(c_i)}$

For a zero coupon bond, $D = t_n$

Consider a bond that pays a total of $n$ coupons at a frequency of $m$ payments a year with $R = F$. Let the nominal bond yield and nominal coupon rate to be $\lambda$ and $c$ respectively, then

$$D = \frac{1 + \lambda/m}{\lambda} - \frac{1 + \lambda/m + \frac{n}{m}(c - \lambda)}{\lambda + c[(1 + \lambda/m)^n - 1]}$$

**Duration and Sensitivity of $P - \lambda$**

We have $\frac{dP}{dr} = \left(-\frac{D}{1+r}\right)P$ for any cashflow $C$ with an effective annual rate of $r$

We call the term $\frac{D}{1+r}$ as the **modified duration** $D_M$

The duration for a perpetuity is given as $\frac{1+r}{r}$

For a bond that pays $m$ coupons a year with a nominal bond yield $\lambda$, we have $\frac{dP}{d\lambda} = \left(-\frac{D}{1+\lambda/m}\right)P$

We shall note that $-D_M P$ is the slope of the tangent line to the price-yield curve at the point where the yield is $\lambda$, so we have $P(\lambda + \Delta\lambda) \approx P(\lambda) - (D_M P)\Delta\lambda$

**Duration of Bond Portfolio**

The duration of a bond portfolio consisting of $\alpha_i$ units of bond $i$, assuming that the bonds have a common effective annual yield (or take an average yield), the duration of the portfolio $D_p$ is a weighted average of the duration of each individual bonds.

**Convexity**

The **convexity** of the bond is the quantity $C = \frac{d^2P}{d\lambda^2}/P$

Thus we have shown that $P(\lambda + \Delta\lambda) - P(\lambda) = \Delta P \approx P\left[-D_M\Delta\lambda + \frac{1}{2}C(\Delta\lambda)^2\right]$

## 2.4 Yield Curves and Term Structure of Interest Rates

The relationship between yield and maturity is called the **term structure of interest rates**. The graphical representation is known as a **yield curve**

The yield curve is

1. upward sloping or **normal** if people are optimistic about the future
2. **flat** if people are indifferent about the future
3. downward sloping or **inverted** if people are pessimistic about the future

#### Spot and Forward Rates

**Spot rates** $s_t$ is the annual interest rate that begins today ($t = 0$) and lasts until some future time $t$. In effect, $s_t$ is the yield-to-maturity (YTM) of a zero-coupon bond that matures at time $t$

**Forward rates** $f_{j,k}$ is the interest rate observed at some future time $j > 0$ and lasts until a time $k > j$.

We have $(1 + s_k)^k = (1 + s_j)^j (1 + f_{j,k})^{k-j}$

#### Bootstrap Method

To determine $s_n$ when $(s_1, s_2, \ldots, s_{n-1})$ has been found: find a $n$-year coupon bond with price $P$ and cashflow $(c_1, c_2, \ldots c_n)$ and solve for $s_n$.

# 3 Expected Utility Theory

## 3.1 + 3.2 Basic Terminologies

Utility function $U$ of a person is real-valued, continuous, and increasing.

Investment decisions (invest, indifferent, avoid) are made based on the expected utility of his final wealth $E[U(W)] = E[U(w_0 + X)]$, where $w_0$ is his initial wealth and $X$ the random payoff of a risky prospect, (>, = , <) w.r.t. $U(w_0)$

#### Concave and Convex Utility Functions

A function $U$ defined on an open interval $I$ is **concave** if for any given $\alpha \in [0, 1]$ and for any $x, y \in I$, $U(\alpha x + (1-\alpha)y) \geq \alpha U(x) + (1-\alpha)U(y)$

A function $U$ defined on an open interval $I$ is **convex** if for any given $\alpha \in [0, 1]$ and for any $x, y \in I$, $U(\alpha x + (1-\alpha)y) \leq \alpha U(x) + (1-\alpha)U(y)$

A linear function is both convex and concave.

**Jensen's Inequality**. Let $X$ be a random variable and let $U$ be a strictly concave function. Then, $E[U(X)] < U(E[X])$. Reverse inequality signs if $U$ is strictly convex.

**Risk Attitude**. An individual with utility function $U$ is (1) risk averse iff $U$ is strictly concave, (2) risk neutral iff $U$ is linear, (3) risk loving iff $U$ is strictly convex.

Observe: a risk-averse individual will avoid fair game, i.e. $E[X] = 0$

The **certainty equivalent** of a risky prospect with payoff $X$ w.r.t. a utility function $U$ is the real number $c = CE(X; U)$ where $U(c) = E[U(w_0 + X)]$

The **risk premium** of a risky prospect with payoff $X$ w.r.t. a utility function $U$ is the real number $RP(X;U) = w_0 - CE(X;U)$

**Positive Affine Transformation**. Let $U$ be an utility function. For any $\alpha > 0, \beta \in \mathbb{R}$, $\alpha U + \beta$ is a positive affine transformation of $U$.

Corollary:

1. Two investors whose utility functions are affine transformations of one another have the same risk attitude.
2. $CE(X; \alpha U + \beta) = CE(X;U)$ AND $RP(X; \alpha U + \beta) = RP(X;U)$

## 3.3 Arrow-Pratt Measures of Risk Aversion

For a **risk averse** individual with utility function $U$, his **Arrow-Pratt absolute risk aversion** (ARA) coefficient at wealth level $w$ is $U_{\text{ARA}}(w) = -\frac{U''(w)}{U'(w)} > 0$

Result. $U_{\text{ARA}} = V_{\text{ARA}}$ iff $U$ and $V$ are positive affine transformations of each other.

We say 2 utility functions are **equivalent** iff they have the same ARA.

We use ARA to compare the degree of risk aversion of 2 individuals. If $U_{\text{ARA}} > V_{\text{ARA}}$ for all $w$, we say that the individual with utility function $U$ is **globally more risk averse** than the individual with utility function $V$.

We can restate the above result as follows:
**Theorem**. An individual with utility function $U$ is **globally more risk averse** than an individual with utility function $V$ iff exists an increasing and strictly concave function $g$ s.t.
$U(w) = g(V(w))$

Definition. Arrow-Pratt Relative Risk Aversion (RRA) $U_{\text{RRA}}(w) = -wU_{\text{ARA}}(w)$

## 3.4 Portfolio Selection

An individual with initial wealth $w_0$ can invest a portion $\alpha w_0$ where $\alpha \in [0,1]$ of his money in a risky investment $X$ with a random **rate of return** $R$. Then the expected utility of his final wealth is $E[U(W)] = E[U(w_0(1+\alpha R))]$.

The individual will seek $\alpha$ that maximises $E[U(W)]$

# 4 + 5 Portfolio Theory

## n risky assets

**Rate of Return of Asset**. If $W_0$ is invested in an asset at time $t = 0$ is worth a random amount of $W_1$ at time $t = 1$, then the rate of return of asset is a random variable $r = \frac{W_1 - W_0}{W_0}$

**Risk of Asset**. A measure of the risk of asset $i$ is $\sigma_i = \sqrt{\text{Var}(r_i)}$.

**Correlation of returns**. A statistical measure of the returns of two assets $i$ and $j$ is the covariance $\sigma_{i,j} = \text{cov}(r_i, r_j)$.

The correlation coefficient $\rho_{i,j} = \sigma_{i,j}/(\sigma_i\sigma_j)$.

**Short Selling.** An individual shorts an asset by borrowing a certain number of units of asset at $t = 0$ to sell immediately at $W_0$ to which he would buy the same number of units of the asset at some pre-agreed date $t = 1$ at $W_1$. The borrower profits $W_0 - W_1$ (limited profit, unlimited loss)

Notation. We can denote shorting as a negative number of units of asset.

**Assumption**. the covariance matrix $C$ is positive definite, i.e. for all nonzero vector $\mathbf{x}$, we have $\mathbf{x^T C x} > 0$. (Fact: $C$ is positive semi-definite)

**Definition**.

- Let the vector $w = (w_1, w_2, \ldots, w_n)^T$ be an individual's portfolio that sum to 1.
- The rate of return $r_p$ of the portfolio is defined as the weighted sum of the rate of return of the individual assets, i.e. $r_p = \sum_{i=1}^{n} w_i r_i$
- The **portfolio mean** $\mu_p = E[r_p] = \sum_{i=1}^{n} w_i \mu_i$ or $\mu_p = \mathbf{w}^T \mu$ where $\mu$ is the **mean vector**
- The **portfolio variance** $\sigma_p^2 = \mathrm{Var}(r_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j}$ or $\sigma_p^2 = \mathbf{w^T C w}$ where $\mathbf{C}$ is the **covariance matrix**
- $\mathbf{C}, \mathbf{C^{-1}}$ are symmetric and **positive definite**, i.e. for any non-zero vector $\mathbf{x}$, we have $\mathbf{x^T C x} > 0$
- For any two portfolios with return rates $r_1, r_2$ and weight vectors $\mathbf{w_1, w_2}$ respectively, we have $\mathrm{cov}(r_1, r_2) = \mathbf{w_1^T C w_2}$
- $\frac{d}{d\mathbf{w}}(\sigma_p^2) = 2\mathbf{Cw}$

**Minimum-Variance Frontier & Global Minimum-Variance Portfolio** (GMVP)

> Problem 1. Fix $\mu$. We seek a portfolio of minimum $\sigma_p$ s.t. $\mu_p = \mu$. Such portfolio lies on the **minimum-variance frontier.**

Let $a = \mathbf{1^T C^{-1} 1}, b = \mathbf{1^T C^{-1}}\mu, c = \mu^\mathbf{T} \mathbf{C^{-1}} \mu$

Since $\mathbf{C}$ is positive definite, we have $a, b > 0$. Moreover, using Cauchy-Schwartz Inequality we have $\Delta = ac - b^2 > 0$

Given any $\mu_p$, the minimum-variance portfolio is given as

$$w_p = \frac{c - b\mu_p}{ac - b^2}\mathbf{C^{-1} 1} + \frac{a\mu_p - b}{ac - b^2}\mathbf{C^{-1}}\mu$$

The minimum-variance frontier curve ($\mu_p$ against $\sigma_p$) is defined as

$$\sigma_p^2 = \frac{a\mu_p^2 - 2b\mu_p + c}{ac - b^2} = \frac{a}{ac - b^2}\left(\mu_p - \frac{b}{a}\right)^2 + \frac{1}{a}$$

> Problem 2. Find a portfolio with the minimum portfolio risk $\sigma_p$. Such portfolio is called the global minimum-variance portfolio (**GMVP**).

GMVP satisfies the following:

- $w_{\text{GMVP}} = \frac{\mathbf{C^{-1}1}}{\mathbf{1^TC^{-1}1}}$
- $\mu_{\text{GMVP}} = \frac{b}{a}$ and $\sigma^2_{\text{GMVP}} = \frac{1}{a} = \text{cov}(r_p, r_{\text{GMVP}})$ for any portfolio $p$.
- $\text{cov}(r_p, r_{\text{GMVP}}) = \sigma^2_{\text{GMVP}}$ for any portfolio in **risky** assets $p$

The upper half of the minimum-variance frontier, above GMVP, is called the **efficient frontier**. I.o.w., a portfolio $(\sigma_p, \mu_p)$ lying on the frontier is efficient iff $\mu_p > \frac{b}{a}$

**Feasible Sets**. The feasible sets for 2 asset = minimum variance frontier. Given any 2 risky assets 1 and 2, the feasible set is

1. A straight line joining $(\sigma_1, \mu_1)$ and $(\sigma_2, \mu_2)$ if $\rho_{1,2} = 1$
2. A V-shaped graph comprising two straight lines, each joining the $(\sigma, \mu)$ of 1 asset to a point with zero portfolio variance when $\rho_{1,2} = 1$
3. A curve passing through $(\sigma_1, \mu_1)$ and $(\sigma_2, \mu_2)$ if $|\rho_{1,2}| < 1$

When there are $n > 2$ assets, the feasible set is the solid region enclosed by all the curves formed by portfolios with 2 assets.

**Two Fund Theorem**. Let $w_1$ and $w_2$ be the weight vectors of any 2 distinct portfolios on the minimum-variance frontier. Then, the minimum-variance set of portfolios is the set $\{\alpha w_1 + (1-\alpha)w_2 \mid \alpha \in \mathbb{R}\}$

Corollary. Let $w_1 = \frac{\mathbf{C^{-1}1}}{\mathbf{1^TC^{-1}1}}$ be the GMVP, and $w_2 = \frac{\mathbf{C^{-1}}\mu}{\mathbf{1^TC^{-1}}\mu}$.

- $\mu_2 = \frac{c}{b}$ and $\sigma_2^2 = \frac{c}{b^2}$
- $w_2$ is efficient if $b = \mathbf{1^TC^{-1}}\mu > 0$.
- $\alpha w_1 + (1-\alpha)w_2$ is efficient for all $\alpha < 1$

Tutorial 6.
Given an initial wealth $w_0$ and two uncorrelated investments $X_1$ and $X_2$ with rates of returns $r_1$ and $r_2$ respectively, where $r_i \sim N(\mu_i, \sigma_i^2)$, a risk averse investor whose utility function is any positive affine transformation of $U(x) = -e^{-\lambda x}$, $\lambda > 0$ allocates $\alpha W_0$ in $X_1$ and $(1-\alpha)W_0$ in $X_2$ where $\alpha \in [0,1]$ s.t. $f(\alpha) = \alpha\mu_1 + (1-\alpha)\mu_2 - \frac{\lambda w_0}{2}(\alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2)$ is maximized.

# n risky assets + 1 risk-free asset

Consider a portfolio comprising of the risk free asset with a deterministic rate of return $r_f$ (also called the **risk-free rate**) and the $n$ risky assets.

Define.

- Let the vector $w = (w_1, w_2, \dots, w_n)^T$ be the weight vector for the portfolio in the $n$ risky assets. However, the assumption $\sum\limits_{i=1}^{n} w_i = 1$ might no longer hold.
- The rate of return of the portfolio $r_p = (1 - \mathbf{1^Tw})r_f + \mathbf{w^Tr}$ where $\mathbf{r}$ is the vector of rates of return of risky assets.
- Let $\mu$ be the mean vector. Then, the portfolio mean $\mu_p = (1 - \mathbf{1^Tw})r_f + \mathbf{w}^T\mu$
- The portfolio variance is $\sigma_p^2 = \mathbf{w^TCw}$

> Problem 3. Problem 1 in the $n$ risky assets + 1 risk-free asset context.

Given any $\mu_p$, the minimum-variance portfolio is given as

$$\mathbf{w}_p = \frac{(\mu_p - r_f)\mathbf{C}^{-1}(\mu - r_f\mathbf{1})}{(\mu - r_f\mathbf{1})^T\mathbf{C}^{-1}(\mu - r_f\mathbf{1})}$$

The minimum-variance frontier curve ($\mu$ against $\sigma$) is defined as

$$\sigma_p^2 = \frac{(\mu_p - r_f)^2}{(\mu - r_f\mathbf{1})^T\mathbf{C}^{-1}(\mu - r_f\mathbf{1})} = \frac{(\mu_p - r_f)^2}{c - 2br_f + ar_f^2}$$

All minimum-variance portfolios lies on 2 rays emanating from the point $(0, r_f)$. The efficient frontier is known as the **Capital Market Line** (CML). The feasible set lie within the open triangular region enclosed by the two region.

We shall now care more for the case when $\mu_{\text{GMVP}} = \frac{b}{a} > r_f$, i.e. investing in risky assets have a slightly larger rate of return compared to risk-free assets. This is natural in the real world, otherwise you wouldn't want to invest in a risky asset.

The **tangency portfolio** is the unique portfolio that lies on both the CML and the minimum-variance frontier for risky assets, i.e. it invests only in the $n$ risky assets. Then, we have $\mathbf{w}_{\tan} = \frac{\mathbf{C}^{-1}(\mu - r_f\mathbf{1})}{\mathbf{1}^T\mathbf{C}^{-1}(\mu - r_f\mathbf{1})}$, $\mu_{\tan} = \frac{c - r_f b}{b - r_f a}$ and $\sigma_{\tan}^2 = \frac{c - 2r_f b + ar_f^2}{(b - r_f a)^2}$

As such, the CML can be expressed as $\mu_p = r_f + \frac{\sigma_p}{\sigma_{\tan}}(\mu_{\tan} - r_f)$

**Sharpe Ratio.** $SR_p = (\mu_p - r_f)/\sigma_p$ is the risk-adjusted return. Portfolios that lie on the CML attain the highest Sharpe ratio.

**One-Fund Theorem.** In a financial market with risky assets and a risk-free asset, an investor will choose to hold only the risk-free asset and the tangency portfolio. Investors differ only in the proportion of total wealth allocated to the tangency portfolio.

(Intuitively, One-Fund Theorem says that rational investors only invests along the CML; the proportion allocated to the tangency portfolio is determined by an investor's risk-averseness)

Consequence. all investors hold risky assets in the same relative proportion, that is as given by the tangency portfolio.

## 5.4 (Capital Asset Pricing Model) CAPM

**Market portfolio.** Suppose there are $u_i$ units of asset $i$, priced at $p_i$ per unit, the total market cap of asset $i$ is $u_i p_i$. The market portfolio $w_m = (w_1, w_2, \ldots, w_n)^T$ is given by $w_i = \frac{u_i p_i}{\sum_{i=1}^{n} u_i p_i}$

In CAPM, assuming demand = supply, tangency portfolio = market portfolio.

**Beta of Assets** We call $\frac{\sigma_{i,m}}{\sigma_m^2}$ as the beta of asset $i$, denoted by $\beta_i$.

**CAPM Theorem.** For any portfolio $p$ investing in $n$ risky assets and 1 risk free asset, we have $\mu_p - r_f = \beta_p(\mu_m - r_f)$ where $\beta_p = \text{cov}(r_p, r_m)/\sigma_m^2$.

- $\mu_m - r_f$ is the **excess return** (expected return in excess of the risk-free rate) of market portfolio

**Portfolio beta.** $\beta_p = \mathbf{w^T}\beta$, where $\beta$ is the beta vector with $\beta_i = \sigma_{i,m}/\sigma_m^2$, measures the sensitivity of excess return of $p$ w.r.t. excess return of the market portfolio. Clearly $\mathbf{w^T}\beta = \beta_m = 1$ and $\beta_f = 0$

For any efficient portfolio $p$ lying on the CML:

1. the CAPM coincides with the equation of the CML, i.e. correlation = 1 and $\text{cov}(r_p, r_m) = \sigma_p\sigma_m$
2. From one fund theorem, let $r_p = \alpha r_m + (1-\alpha)r_f$. Then, $\beta_p = \alpha$ and $\sigma_p^2 = \beta_p^2\sigma_m^2$

For 2 portfolios $i$ and $j$, CAPM gives $\frac{\mu_i - r_f}{\mu_j - r_f} = \frac{\beta_i}{\beta_j}$
Corollary. $\mu_i = \mu_j$ iff $\beta_i = \beta_j$.

Tutorial 10. Let $p$ be any portfolio and let $q$ be a portfolio on the CML. Then $\beta_{pq} = \text{cov}(r_p, r_q)/\sigma_q^2 = \beta_p/\beta_q$ and $\mu_p - r_f = \beta_{pq}(\mu_q - r_f)$

## Security Market Line

The relationship between mean return and beta for any asset is a straight line (called the SML) of slope $\mu_m - r_f$.

Within the framework of CAPM, the return of any asset should fall on the SML. Thus, we can call a portfolio undervalued (fairly valued, overvalued) if its estimated return is > ( = , <) the benchmark return given by CAPM.

The difference between the estimated return and the benchmark return is called **alpha** of the return. Undervalued assets have positive alphas.

## Systematic and Non-systematic Risk

The CAPM suggests that $r_p = r_f + \beta_p(r_m - r_f) + \epsilon_p$ for some random variable $\epsilon_p$. It is easy to see that $E[\epsilon_p] = 0$ and $\text{cov}(r_m, \epsilon_p) = 0$.

It follows that $\sigma_p^2 = \beta_p^2\sigma_m^2 + \text{Var}(\epsilon_p)$. The total risk is a sum of

1. **systematic risk** $\beta_p^2\sigma_m^2$ : market risk that cannot be diversified
2. **non-systematic risk** (specific risk / idiosyncratic risk) $\text{Var}(\epsilon_p)$ which is not correlated to the market and hence can be reduced via diversification.

**Theorem.** Under the assumptions of the CAPM, an asset or portfolio carries only systematic risk if and only if it lies on the CML (i.e. efficient)

# 6 Basic Option Theory

Some definitions

1. Derivatives are financial instruments whose value depends on some underlying assets, such as stocks and bonds. An option is a derivative
2. A **call (put) option** is a contract that gives its holder the right to buy (sell) a specific quantity of an underlying asset at a specified price **(strike / exercise price)**, on or before a specified date **(expiration/maturity date)**
3. In an option contract, the party that holds the right is the buyer/holder of the contract and is said to be in the **long position**. The party on the other side of the contract is called the seller/writer of the contract and is said to be in the **short position**
4. American options can be exercised anytime before the expiration date while European options can only be exercised on the expiration date.
5. The holder of an option pays the writer an up-front fee known as the **option premium/price**

The following chapter only discusses the European option.

Notation.

- $T$: maturity date of option
- $K$ : strike price
- $S_t$: price of underlying asset at time $t$
- $C$: current call option price
- $P$: current put option price

**Moneyness of options.** An option is said to be out-of-the-money (at-the-money, in-the-money) if it has a negative (0, positive) payoff.

### Payoff and Profit

1. The payoff from a long position in a call option is $\max(S_T - K, 0)$
2. The payoff from a long position in a put option is $\max(K - S_T, 0)$
3. The payoff from a short position in an option is the negative of the payoff from a long position in the option.
4. The profit of an option is the option's payoff - the time value at $T$ of the option price.

Remarks: The writer of a call option face unlimited downsides.

## 6.3 Options Trading Strategy

All options here are for 1 asset.

1. Covered Calls: long 1 asset + short 1 K-call
2. Protective Puts: long 1 asset + long 1 K-put
3. Bull Spreads: long 1 $K_1$ option + short 1 $K_2$ option where $K_1 < K_2$
4. Symmetric Buttefly Spreads: long 1 $K_1$ option + short 2 $K_2$ option + long 1 $K_3$ option where $K_1 < K_3$ and $K_2 = (K_1 + K_3)/2$
5. Long (Short) straddles: long (short) 1 K-call + long (short) 1 K-put

## 6.4 Arbitrage-Based Restrictions on Option Prices

**Arbitrage Opportunity.** Let $V(0)$ be the value of a portfolio today (i.e. how much would you pay to get the portfolio today) and $V(t)$ is the value at some future time $t$, then an arbitrage opportunity arises if either:

1. $V(0) = 0$ and $V(T) \geq 0$ with $P[V(t) > 0] > 0$ for some $t > 0$.
2. $V(0) < 0$ and $V(T) \geq 0$.

**No-arbitrage Principle.** Arbitrage opportunities do not exist.

(More like if arbitrage opportunity exists, it will be exploited and market will move back to a no-arbitrage condition)

**Bounds on option prices**

Let $C(K)$ and $P(K)$ be the call and put option values with strike price $K$ respectively.

1. $\max(0, S_0 - Ke^{-rT}) \leq C \leq S_0$
2. $\max(0, Ke^{-rT} - S_0) \leq P \leq Ke^{-rT}$
3. If $K_1 < K_2$, then $0 \leq C(K_1) - C(K_2) \leq e^{-rT}(K_2 - K_1)$
4. If $K_1 < K_2$, then $0 \leq P(K_2) - P(K_1) \leq e^{-rT}(K_2 - K_1)$
5. For any $\alpha \in (0,1)$, $P(\alpha K_1 + (1-\alpha)K_2) \leq \alpha P(K_1) + (1-\alpha)P(K_2)$, i.e. $P$ is convex.

Result. When the underlying asset pays no dividends, it is never optimal to exercise an American call option early.

**Put-Call Parity**. Only for European options. Suppose a put option with price $P$ and a call option with price $C$ are written on the same non-dividend paying asset, have the same strike price $K$ and have the same maturity $T$, then $C + Ke^{-rT} = P + S_0$

No-arbitrage argument for equality. Suppose we have two portfolios $U$ and $V$ s.t. $U(T) = V(T)$, then $U(0) = V(0)$.

## 6.5 Binomial Option Pricing

**The Binomial Model** : Cox-Ross-Rubinstein (CRR) Model

We consider an option on an underlying asset which trades during a fixed time interval $[0, T]$, divided into $N$ equally-spaced sub-intervals $[t_{i-1}, t_i]$ where $i = 1, 2, \ldots, N$. Let $S_i$ be the asset price at time $t_i$. Let $\delta = T/N$

Under the binomial model, $S_i$ depends on $S_{i-1}$ and have 2 outcomes:

1. $S_i = S_i^u = uS_{i-1}$ with probability $p$
2. $S_i = S_i^d = dS_{i-1}$ with probability $1-p$

for some constants $d, u$ satisfying $0 < d < 1 < u$.

We assume a continuously compounded risk-free rate $r$ satisfying $d < e^{r\delta} < u$. The inequality ensures there are no arbitrage opportunities.

Let $F_i$ be the value of an option at time $t_i$. For a one-step binomial model, i.e. $N = 1$, we can construct a **replicating portfolio** $(\Delta, B)$ consisting of $\Delta$ units of the underlying asset + $B$

amount invested in a risk-free asset such that it has the same end-of-period value as the option. We have $\Delta = \frac{F_1^u - F_1^d}{S_0(u-d)}$ and $B = \frac{uF_1^d - dF_1^u}{e^{r\delta}(u-d)}$

Based on the no-arbitrage principle, the initial value of the portfolio is equalt to that of the option, i.e. $\Delta S_0 + B = F_0$

Call $q = \frac{e^{r\delta} - d}{u-d}$ the **risk-neutral probability**, then we have $F_0 = e^{-r\delta}E[F_1]$ where $E[F_1] = qF_1^u + (1-q)F_1^d$

Remark:

1. The above option pricing formula is independent of the probability $p$ of asset moving up or down
2. $S_0 = e^{-r\delta}E(S_1)$
3. We can extend the above to a two-step binomial model (apply the one-step binomial method twice to get $F_1^d$ and $F_1^u$) and get $F_0 = e^{-2r\delta}(q^2F_2^{uu} + q(1-q)(F_2^{ud} + F_2^{du}) + (1-q)^2F_2^{dd})$